18.152 PROBLEM SET 5

due April 25th 9:30 am.

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. (a) *Let* $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = u_{xx} && x \in \mathbb{R}, t \geq 0 \\ &u(x, 0) = g(x), u_t(x, 0) = h(x) && x \in \mathbb{R}, \end{aligned}$$

where g, h *are smooth. Suppose that there exists some large constant* R *such that* $g(x) = h(x) = 0$ *for* $|x| \geq R$*. Show that there exists some large constant* T *depending on* R *such that*

$$\int_{\mathbb{R}} |u_t(x,t)|^2 dx = \int_{\mathbb{R}} |u_x(x,t)|^2 dx, \tag{1}$$

holds for $t \geq T$.

(b) *Give an example of a solution to the wave equation on the bounded domain* $(x, t) \in [0, L] \times [0, +\infty)$ *such that it has constant Dirichlet data and does not satisfies* (1).

Hint: Use the d'Alembert formula.

Problem 2. *Let* $u : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}$ *be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = \Delta u && x \in \mathbb{R}^3, t \geq 0 \\ &u(x, 0) = g(x), u_t(x, 0) = h(x) && x \in \mathbb{R}^3, \end{aligned}$$

where g, h *are smooth. Suppose that there exists some large constant* R *such that* $g(x) = h(x) = 0$ *for* $|x| \geq R$*. Show that there exists some large constant* C *depending on* $R, g, \nabla g, h$ *such that*

$$|u(x,t)| \leq C/t$$

holds for $t > 0$.

Hint: Use the Kirchhoff's formula.

Problem 3. *Let* $u : \mathbb{R}^6 \times [0, +\infty) \to \mathbb{R}$ *be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = \Delta u && x \in \mathbb{R}^6, t \geq 0 \\ &u(x,0) = g(x), u_t(x,0) = h(x) && x \in \mathbb{R}^6, \end{aligned}$$

where g, h *are smooth. Suppose that there exists some large constant* R *such that* $g(x) = h(x) = 0$ *for* $|x| \geq R$. *Establish a bound for* $|u(x,t)|$ *in terms of* t, R, g, h *and the first or higher order derivatives of* g, h.

Hint: Use some energy.

Problem 4. *Let* $u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ *be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = \Delta u && x \in \mathbb{R}^n, t \geq 0 \\ &u(x,0) = g(x), u_t(x,0) = h(x) && x \in \mathbb{R}^n, \end{aligned}$$

where g, h *are smooth. Suppose that* g, h *are polynomials; for example,* $g(x_1, x_2, x_3) = x_1^2 - 2x_2x_3 + 3$. *Show that the solution* $u(x,t)$ *is also polynomial for each* $t \geq 0$.

Hint: Use some energy.